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Hamiltonian circle actions on symplectic manifolds and the signature

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Abstract

Let M be a symplectic manifold with a Hamiltonian circle action with isolated fixed points. We prove that $\sigma(M) = b_0(M) - b_2(M) + b_4(M) - b_6(M) + \cdots$ where $\sigma(M)$ is the signature of M and $b_i(M)$ is the ith Betti number of M.

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1. Introduction

A smooth action of a connected Lie group G on a symplectic manifold (M, ω) is *Hamiltonian* if there is a map $\mu: M \to \mathfrak{g}^*$ where $\mathfrak{g}^*$ is the dual of the Lie algebra $\mathfrak{g} = T_1 G$ of G, with the following properties:

- The map μ is equivariant with respect to the given action of G on M and the coadjoint action of G on $\mathfrak{g}^*$.
- for any $\xi \in \mathfrak{g}$ let V_ξ be the vector field on M defined by the action of G and the element $\xi \in \mathfrak{g}$. Then

  $$\omega(V_\xi, X) = \mathrm{d}\mu(X)(\xi)$$

  for any vector field X on M.

The map μ is called a *momentum map* for the action of G. The purpose of this note is to prove the following theorem.

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Theorem 1.1. *Let M be a compact symplectic manifold with a Hamiltonian action of the circle which has isolated fixed points. Then*

$$\sigma(M) = \sum b_{4i}(M) - \sum b_{4i+2}(M),$$

where $\sigma(M)$ is the signature of M and $b_j(M)$ is the jth Betti number of M.

The proof of this theorem is not difficult. First, the Atiyah–Bott fixed point theorem for the signature operator of M, see [1,2], shows that

$$\sigma(M) = \sum_p \eta(p),$$

where the sum is over the fixed points of the circle action, and for each fixed point p, $\eta(p)$ is computed from local data. Next, since the circle action is Hamiltonian, its fixed points are the critical points of the momentum map $\mu: M \to \mathbb{R}$. The critical points of the momentum map are all non-degenerate and the Morse index $\operatorname{ind}(p)$ of each critical point p is even. We show that, in the above formula for $\sigma(M)$,

$$\eta(p) = (-1)^{\operatorname{ind}(p)/2}.$$

Since the Morse index of each critical point is even, the number of critical points of index $2k$ is $b_{2k}(M)$ and this proves the theorem. The detailed proof is given in Sections 2 and 3.

We discuss an application of this result to symplectic 4-manifolds in Section 4, and also use the method of proof to show that if a symplectic 4-manifold has non-zero signature then any non-trivial Hamiltonian action of the circle must have at least one isolated fixed point.

2. The fixed point formula for the signature

Suppose M is a compact manifold and

$$D: C^\infty(M; E) \to C^\infty(M; F)$$

is an elliptic differential operator mapping (smooth) sections of a vector bundle E over M to (smooth) sections of a vector bundle F over M. Then ellipticity implies that both the kernel of D and the cokernel of D are finite-dimensional. The index of D is defined by

$$\operatorname{ind} D = \dim \ker D - \dim \operatorname{coker} D.$$

Now suppose we have a compact group G acting compatibly on M, E, and F. Then G acts on both $C^\infty(M; E)$ and $C^\infty(M; F)$. Suppose, in addition, that D commutes with this action of G. Then both $\ker D$ and $\operatorname{coker} D$ are representations of G and, by definition, the character valued index of D is the character of the virtual representation $\ker D - \operatorname{coker} D$ of G. Explicitly, this character valued index is the class function

$$\operatorname{ind}\,(g; D) = \operatorname{tr}(g|_{\ker D}) - \operatorname{tr}(g|_{\operatorname{coker} D})$$

on G.

The Atiyah–Bott fixed point theorem gives a formula for this character valued index in terms of the fixed point set of the action of G on M. We need the detailed formula in the case of the signature operator on a compact manifold equipped with an action of the circle which has isolated fixed points.

Suppose M is a $2n$-dimensional closed oriented Riemannian manifold. Then we have the operator

$$\varepsilon = i^{p(p-1)+n} * : \Omega^p(M) \to \Omega^{2n-p}(M),$$

where $\Omega^p(M)$ denotes complex valued p-forms on M and $*$ is the Hodge star operator. Now $\varepsilon^2 = 1$ and so we define

$$\Omega^+(M), \quad \Omega^-(M)$$

to be the $+1$ and -1 eigenspaces of ε. The operator $d + d^*$, where d is the exterior derivative and $d^* = - * d *$, maps the $+1$ eigenspace of ε to the -1 eigenspace and so defines an operator

$$A : \Omega^+(M) \to \Omega^-(M).$$

This is an elliptic operator – the *signature operator*.

Now suppose S^1 acts on M by isometries. Then the operator A commutes with the induced actions of S^1 on $\Omega^\pm(M)$. Suppose further that the fixed points of the S^1 action on M are isolated. Then the tangent space $T_p(M)$ of M at a fixed point p is a representation of the circle. Since p is an isolated fixed point, the trivial representation cannot be a summand in $T_p(M)$ and so

$$T_p(M) \cong E_1 \oplus \cdots \oplus E_n,$$

where each E_i is an irreducible two-dimensional real representation of the circle. Choose an identification of E_i with $\mathbb{C}$ so that $z \in S^1$ acts on E_i by multiplication by $z^{m_i(p)}$ with $m_i(p) > 0$. The integers $m_i(p)$ are the exponents of the circle action at p. Define $\eta(p)$ to be ± 1 according to whether the orientation of M at p agrees with the direct sum of the complex orientations of E_i or not. Then, see [1,2], the fixed point formula gives

$$\sigma(z; M) = \operatorname{ind}(z; A) = \sum_p \eta(p) \prod_{i=1}^{n} \frac{1 + z^{m_i(p)}}{1 - z^{m_i(p)}}$$

for those elements $z \in S^1$ whose fixed point set is the same as the fixed point set of the whole circle. Here $\sigma(z; M)$ is the character valued index of the signature operator on M, or more briefly the character valued signature. The condition on z is satisfied by any element in S^1 of infinite order, thus it is satisfied on an open dense subset of S^1.

Lemma 2.1. *Let M be a compact oriented Riemannian manifold. Suppose the circle acts on M by isometries and has isolated fixed points. Then the character valued signature $\sigma(z; M)$ is independent of z and*

$$\sigma(z; M) = \sum_p \eta(p),$$

where the sum is taken over the fixed point set of the circle action.

Proof. To prove this lemma we use an argument of Bott and Taubes [4]. The character valued signature $\sigma(z; M)$ is the character of a finite-dimensional virtual representation of S^1; therefore it is given by finite Laurent series and so it extends to a unique meromorphic function on $\mathbb{C} \cup \infty$ which can only have poles at 0 and ∞. On the other hand

$$\sum_p \eta(p) \prod_{i=1}^n \frac{1 + z^{m_i(p)}}{1 - z^{m_i(p)}}$$

is a rational function which can only have poles on the unit circle. By the fixed point formula, this rational function is equal to $\sigma(z; M)$ on an open dense subset of the circle and hence the two must be equal on the whole of $\mathbb{C} \cup \infty$. Therefore this rational function can only have poles at 0 and ∞; since neither of these lie on the unit circle it has no poles and so must be constant.

To determine this constant we evaluate the right-hand side of the fixed point formula at $z = 0$ and this shows that

$$\sigma(z; M) = \sum_p \eta(p),$$

where the sum is taken over the fixed points of the circle action. □

The usual argument from index theory identifies the kernel and cokernel of the signature operator with subspaces of the cohomology $H^*(M; \mathbb{C})$. Since the circle is connected it must act trivially on cohomology. This can be used to prove that the character valued signature is constant without any hypotheses on the fixed points. However, this line of argument does not give the value of the constant. The fact that this constant is $\Sigma\eta(p)$ in the case where the circle action has isolated fixed points p is required for the proof of Theorem 1.1.

Lemma 2.2. *Let M be a compact oriented manifold and suppose the circle acts on M with isolated fixed points; then*

$$\sigma(M) = \sum_p \eta(p),$$

where the sum is taken over the fixed points of the circle action.

Proof. Choose an invariant Riemannian metric on M. Then the formula follows from Lemma 2.1 since $\sigma(M) = \sigma(1; M)$. □

3. Hamiltonian circle actions with isolated fixed points

Now let M be a compact symplectic manifold of dimension $2n$ with a Hamiltonian action of the circle which has isolated fixed points. Let $\mu : M \to \mathbb{R}$ be a momentum map for this circle action. Then we can draw the following conclusions:

- A point $p \in M$ is a critical point of μ if and only if it is a fixed point of the circle.
- The critical points of μ are all non-degenerate and all have even index.
- If p is a critical point of μ with index $\mathrm{ind}(p)$ then

$$\eta(p) = (-1)^{\mathrm{ind(p)}/2}.$$

The first of these statements follows easily from the conditions the momentum map must satisfy.

To prove the others, we choose a metric g and an almost complex structure J on M which satisfy the usual compatibility hypotheses

$$\omega(X, Y) = g(X, JY) = -g(JX, Y)$$

for all vector fields X, Y on M. Here, of course, ω is the symplectic form on M. Let p be a fixed point of the circle action. The almost complex structure J makes $T_p(M)$ into a complex representation of S^1 and we can decompose this representation into its irreducible summands

$$T_p(M) = E_1' \oplus \cdots \oplus E_n'$$

with exponents $m_i'(p) \in \mathbb{Z}$: thus $z \in S^1$ acts on E_i' by multiplication by $z^{m_i'(p)}$. Note that in this decomposition we cannot assume that $m_i'(p)$ are positive but since p is an isolated fixed point we do know that $m_i'(p) \neq 0$.

It is straightforward to check using the compatibility of g, ω, and J, and the definition of the momentum map that the Hessian of μ is positive definite on the real subspace of $T_p(M)$ spanned by the summands E_i' with exponents $m_i'(p) > 0$ and negative definite on the real subspace of $T_p(M)$ spanned by the summands E_i' with $m_i'(p) < 0$. Since $m_i'(p) \neq 0$ this proves that p is non-degenerate. It also shows that the index of p is twice the number of negative exponents $m_i'(p)$; in particular $\mathrm{ind}(p)$ is even.

The orientation of M is defined by the almost complex structure J. Now considering p as a fixed point of the circle action, it follows that $\eta(p)$ is 1 if there is an even number of negative exponents $m_i'(p)$ and -1 if there is an odd number. This gives the equation

$$\eta(p) = (-1)^{\mathrm{ind(p)}/2}.$$

We can now complete the proof of the main result of this note.

Proof of Theorem 1.1. The critical points of the momentum map $\mu : M \to \mathbb{R}$ are isolated, non-degenerate and each critical point has even index. Therefore the jth Betti number $b_j(M)$ is equal to the number of critical points of index j. Now Lemma 2.2 shows that

$$\sigma(M) = \sum_p \eta(p)$$

and since

$$\eta(p) = (-1)^{\mathrm{ind(p)}/2}$$

this gives the required formula

$$\sigma(M) = \sum b_{4i}(M) - \sum b_{4i+2}(M). \qquad \square$$

4. Hamiltonian circle actions on symplectic 4-manifolds

The first interesting case of Theorem 1.1 is when M has dimension 4.

Corollary 4.1. *Let M be a compact symplectic 4-manifold with a Hamiltonian action of the circle with isolated fixed points. Then $b_2^+(M) = 1$ where $b_2^+(M)$ is the dimension of the subspace of $H^2(M; \mathbb{R})$ on which the intersection form of M is positive definite.*

Proof. The main theorem shows that

$$\begin{aligned} b_2^+(M) - b_2^-(M) &= b_0(M) - b_2(M) + b_4(M) \\ &= 1 - b_2^+(M) - b_2^-(M) + 1 \end{aligned}$$

and therefore $b_2^+(M) = 1$. $\square$

This Corollary is consistent with the classification of Hamiltonian circle actions on symplectic 4-manifolds due to Karshon and others.

We get another result about circle actions on 4-manifolds by applying the Atiyah–Bott fixed point theorem.

Theorem 4.2. *Let M be a compact 4-manifold with a non-trivial circle action. If the circle has no isolated fixed points then $\sigma(M) = 0$.*

The proof of this theorem uses the full version of the fixed point formula; see [3] for the derivation of this formula. Let M be a closed oriented $2n$-manifold with an action of the circle. It follows that the fixed point set of M is a disjoint union of oriented manifolds and that

$$\sigma(M) = \sum_P \eta(P)\sigma(P),$$

where the sum runs over the components of the fixed point set, $\eta(P)$ is a sign which depends on orientation conventions, and $\sigma(P)$ is the signature of P (with the convention that if P has dimension not divisible by 4 then $\sigma(P) = 0$).

Applying this formula to a 4-manifold with a non-trivial circle action which has no isolated fixed points gives $\sigma(M) = 0$ since the other components of the fixed point set must have dimension 1, 2 or 3.

References

[1] M.F. Atiyah and R. Bott, A Lefschetz fixed-point formula for elliptic complexes I, Ann. of Math. 86 (1967) 374–407.
[2] M.F. Atiyah and R. Bott, A Lefschetz fixed-point formula for elliptic complexes II: Applications, Ann. of Math. 88 (1968) 451–491.
[3] M.F. Atiyah and I.M. Singer, The index of elliptic operators III, Ann. of Math. 87 (1968) 546–604.
[4] R. Bott and C.H. Taubes, On the rigidity theorems of Witten, J. Amer. Math. Soc. 2 (1989) 137–186.